THE FOLLOWING INFORMATION MAY BE USEFUL

\[ h = 1.05 \times 10^{-34} \text{ J s}, \quad s = 6.6 \times 10^{-26} \text{ eV s}, \quad 1 \text{ eV} = 1.60 \times 10^{-19} \text{ J} \]

Electron mass \( m_e = 9.11 \times 10^{-31} \text{ kg} \), Neutron mass \( m_n = 1.67 \times 10^{-27} \text{ kg} \)

\[ k_B = 1.381 \times 10^{-23} \text{ J/K}, \quad N_A = 6.022 \times 10^{23} \text{ mol}^{-1} \]

\[ e^{i\omega t} = \cos(\omega t) + is\sin(\omega t), \quad \cos(\omega t) = \frac{1}{2}(e^{i\omega t} + e^{-i\omega t}), \quad \sin(\omega t) = \frac{1}{2i}(e^{i\omega t} - e^{-i\omega t}) \]

\[ e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots \quad \int_0^\infty x^k e^{-\alpha x} \frac{dx}{\alpha} = \frac{1}{\alpha^{k+1}} \quad \int_0^\infty x^k e^{-\alpha x} \frac{dx}{\alpha} = \frac{k!}{\alpha^{k+1}} \]

\[ f \cdot dA = \int (\nabla \times f) \cdot dA, \quad \nabla \times \nabla g = 0 \]

\[ j_x = -\frac{i\hbar}{2m} \left[ \psi^* \frac{\partial}{\partial x} \psi - \psi \frac{\partial}{\partial x} \psi^* \right] \]

\[ \langle n_e \rangle_{FD} = \frac{1}{\exp(e/k_BT) + 1} \quad \langle n_e \rangle_{BE} = \frac{1}{\exp(e/k_BT) - 1} \]

\[ \langle E \rangle = -\frac{\partial \ln Z}{\partial \beta} \quad \langle M \rangle = \frac{1}{\beta} \frac{\partial \ln Z}{\partial H} \quad \Gamma = -k_B T \ln Z \quad PV = Nk_BT \]

\[ \Omega(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega s} K(s) ds \quad K(s) = \int_{-\infty}^{\infty} e^{-i\omega s} \Omega(\omega) d\omega \]

There are two questions.
Show all your working and attach extra sheets if needed.
1. (a) In one dimensional diffusion, the flux across an area is given by

\[ J(x, t) = -D \frac{\partial}{\partial x} n(x, t), \]

where \( n(x, t) \) is the concentration of particles at a position \( x \) at time \( t \). Derive the diffusion equation relating the rate of change of \( n \) to the gradient of \( J \). Justify your reasoning.

\[ J(x, t) = \nabla \cdot (n(x, t) \mathbf{v}) \]

The number of particles in \( [x_1, x_2] \) is

\[ \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} dx \ n(x, t) \]

Conservation of mass means

\[ -\frac{d}{dt} \int_{x_1}^{x_2} dx \ n(x, t) = -J(x_1, t) + J(x_2, t) \]

\[ = -J(x_1, t) - \left. J(x, t) \right|_{x_1}^{x_2} + J(x_1, t) - \left. J(x_1, t) \right|_{x_0}^{x_2} \]

\[ = 2 D \frac{\partial}{\partial x} J(x, t) \]

(b) The probability that a molecule with velocity between \( v_1 \) and \( v_1 + dv \) is scattered by a molecule with velocity \( v \) is given by

\[ \int \left[ \frac{1}{4\pi^3} |v_1 - v| \right] \left[ v_1 \right] \left[ D(v) \ dv_1 \ dv_1 \ dv_2 \right] \]

where \( D(v) \) is the Maxwell velocity distribution. Explain the physical meaning of each bracketed term.

\[ \frac{1}{4\pi^3} \] - prob. rate for scattering of particle with vel \( \vec{v}_1 \)

\[ |v_1 - v| \] - magnitude relative vel between incident \& target particles

\[ \sigma_0 \] - prob. of scattering in any unit

\[ D(v) \ dv_1 \ dv_1 \ dv_2 \] - prob. target with vel \( \vec{v} \) in range \( \vec{v}, \vec{v} + d\vec{v} \).
(c) The solution \( n(x,t) \) (representing the number of molecules within a slice between \( x \) and \( x+dx \) at time \( t \)) for a semi-infinite cylinder of air with cross section \( A \) and \( N \) molecules initially located at \( x=0 \) is:

\[
n(x,t) = \frac{N}{A \sqrt{\pi D t}} \exp \left( \frac{-x^2}{4Dt} \right).
\]

Use this to write down an integral expression for the average position \( \langle x \rangle \) at time \( t \). Show, without solving the integral, that \( \langle x \rangle \) is proportional to \( t^{1/2} \).

\[
\langle x \rangle = \int_{-\infty}^{\infty} x \left( \int_{-\infty}^{\infty} e^{-x^2/(4Dt)} \, dx \right) \, \frac{dx}{\sqrt{\pi D t}}
\]

Let \( u = x^{1/2} \). Then \( du = (4Dt)^{-1/2} \, dx \), and

\[
\langle x \rangle = \int_{0}^{\infty} u \left( 4Dt \right)^{-1/2} e^{-u^2} \, du \left( 4Dt \right)^{1/2} \, dt
\]

2. The Boltzmann equation in the relaxation time approximation is given by

\[
\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial x} + \nu \cdot \frac{\partial f}{\partial v} = -\frac{f - f_0}{\tau}.
\]

(a) Show that this equation is satisfied by the equilibrium distribution of velocities given by

\[
f_0(v_x, v_y, v_z) = n \left( \frac{m}{2\pi k_B T} \right)^{3/2} e^{-\frac{m(v_x^2 + v_y^2 + v_z^2)}{2k_B T}}.
\]

At equilibrium, \( \frac{\partial f_0}{\partial t} = 0 \)

If \( \vec{v} \) independent of \( \vec{x} \) (as it must be with no external forces, at equilibrium), \( \frac{\partial}{\partial x} f = 0 \).

Also, no forces \( \Rightarrow \vec{V} = 0 \).

Therefore, \( 0 = -\frac{1}{n} \left( f - f_0 \right) = 0 \).
(b) Consider the Langevin equation

$$\frac{dv}{dt} = -\gamma v + \frac{1}{m} F(t)$$

where $F(t)$ is a rapidly varying, random, force. Write $F(t)$ and $v$ in terms of Fourier integrals, and show how their Fourier coefficients must be related to satisfy the Langevin equation. Use the result

$$\frac{k_B T}{\pi n} \int_{-\infty}^{\infty} \langle v(0)v(s) \rangle e^{-i\omega s} ds = \frac{1}{mn} \frac{\gamma}{\gamma^2 + \omega^2}$$

to derive a form of the Nyquist theorem.

$$\mathcal{L} = \int \mathcal{V}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} \mathcal{V}(\omega) dt$$

$$\mathcal{F}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} \mathcal{V}(\omega) dt$$

$$\Rightarrow -i \omega \mathcal{V}(\omega) = -\gamma \mathcal{V}(\omega) + \frac{\mathcal{F}(\omega)}{\omega}$$

$$\Rightarrow \mathcal{V}(\omega) = \frac{\mathcal{F}(\omega)}{\gamma - i \omega}$$

$$\mathcal{F}(\omega) \mathcal{V}(\omega) \mathcal{V}^*(\omega) = \frac{\mathcal{F}(\omega) \mathcal{V}(\omega)}{\gamma^2 + \omega^2}$$

$$\Rightarrow \frac{\mathcal{F}^*(\omega) \mathcal{F}(\omega)}{\omega^2 (\gamma^2 + \omega^2)} = \frac{k_B T}{(\frac{m}{n})^2} \frac{\gamma}{\gamma^2 + \omega^2}$$

$$\Rightarrow \frac{\mathcal{F}(\omega)}{\mathcal{F}^*(\omega)} = \frac{\gamma}{n} \frac{k_B T}{m}$$