Optics and Lasers - Chapter 4

Gaussian-Spherical Light Beams - Underlying Maths and Physical properties
planes are 1 mm apart,
1 micron wavelength
Huygens’ Principle

Given an incident field distribution over some closed surface,

we will regard each point on that surface as a source for a uniform spherical wave (called a Huygens’ wavelet) which radiates from that point.

The total field at any other point in space can be calculated by summing all the wavelets
Tools that we will need (I):

Let's us write our propagating waves as follows:

\[ u(x, y, z, t) = \Re(\tilde{u}(x, y) e^{i(\omega t - \mathbf{z} \cdot \mathbf{k})}) \]  

(4.1)

\[ 2\pi f \quad 2\pi/\lambda \]

i.e. we have assumed that the wave has a sinusoidal time and space dependence.
Tools that we will need (II):

\[ z = z_0 + L \]

\[ |r - r_0| = 0 \text{ if } \alpha \neq 0, \quad = 1 \text{ if } \alpha = 0 \]

\[ \tilde{u}(\hat{r}) = \frac{j}{\lambda} \int \int \tilde{u}(\hat{r}_0) \frac{1 + \cos \alpha}{2} e^{jk|\hat{r} - \hat{r}_0|} \, dx_0 \, dy_0 \]

where \( \hat{r}_0 = (x_0, y_0, z_0) \) and \( \hat{r} = (x, y, z) \).
Simplifying:

Since $\alpha$ is small, call this 1

Can replace with $L$ since only a small difference

Note: this is a convolution.
What does this mean?...

\[ \tilde{u}(\hat{r}) = \frac{je^{jkL}}{\lambda L} \int \int \tilde{u}(\hat{r}_0) e^{j \frac{k}{2L} [(x-x_0)^2+(y-y_0)^2]} \, dx_0 \, dy_0 \]  

(4.4)

1mm apart,
1µm wavelength

\( x-x_0 \) in microns
planes are 1mm apart,
1 micron wavelength
Paraxial Wave Equation

So now we have an example of a wave, and we have a method of propagating arbitrary waves.

Not all waves satisfy Maxwell’s equations so we should limit our consideration to EM waves that do satisfy the equations. We hope from this to find some waves that we can use.

Finally, we are mostly interested in developing the means to deal with laser beams: these are things with a well defined size and direction.
Real Laser Beam - Intensity
Helmholtz Equation

Maxwell’s Equations:

\[ \nabla^2 E = \frac{1}{c^2} \frac{d^2 E}{dt^2} \quad (4.7) \]

Assume \( E(r,t) \) can be written as \( E(r) e^{-j \omega t} \), then

\[ \nabla^2 E + k^2 E = 0 \quad (4.8) \]

where \( k^2 = (\omega/c)^2 = (2 \pi/\lambda)^2 \) Helmholtz Equation

Two example solutions of H.E. are:

\[ Ae^{-j k z} \quad \text{plane wave} \]
\[ \frac{A}{r} e^{-j k \cdot r} \quad \text{spherical wave} \]

Neither is a good laser beam
Example solution: A spherical wave (Tool III)

\[ \tilde{u}(x, y) = Ae^{-jk\sqrt{R^2 + x^2 + y^2}} = Ae^{-jk\left(R + \left(\frac{x^2 + y^2}{2R}\right) + \ldots \right)} \]

\[ \approx Ae^{-jkRe^{-jk\left(\frac{x^2 + y^2}{2R}\right)}} \]  \hspace{1cm} (4.5)

\[ \phi(x, y) = k\left(\frac{x^2 + y^2}{2R}\right) = \frac{\pi}{R\lambda} (x^2 + y^2) \]  \hspace{1cm} (4.6)
Real Laser Beam - Intensity
Gaussian-Spherical Waves

We want a solution that looks like \( E(r) = E_0(r) e^{-jkz} \)
where \( E_0(r) \) decreases as \( x,y \) increase

Assume solution:

\[
|\tilde{u}(x,y)| \propto e^{-\frac{x^2+y^2}{w^2}}
\]

Normalizing:

\[
|\tilde{u}(x,y)| = \sqrt{\frac{2}{\pi w}} e^{-\frac{x^2+y^2}{w^2}}
\]

Assume transverse phase can be expressed as a spherical wave of some \( R \)
(really a quadratic) - what would a linear variation of phase mean?

\[
\tilde{u}(x,y) = \sqrt{\frac{2}{\pi w}} e^{-\frac{x^2+y^2}{w^2}} e^{-jk \left( \frac{x^2+y^2}{2R} \right)}
\]
Complex Radius of Curvature

Let’s combine the exponents and we can rewrite as:

\[ \tilde{u}(x, y) = \sqrt{2} \frac{1}{\pi w} e^{-\frac{x^2 + y^2}{w^2}} e^{-jk\left(\frac{x^2 + y^2}{2R}\right)} \]

Where we have created a new concept

\[ \frac{1}{\tilde{q}(z)} = \frac{1}{R(z)} - j\frac{\lambda}{\pi w(z)^2} \]
Using Huygens’ Integral

Start with a Gaussian plane wave i.e. $R \Rightarrow \infty$

\[
\frac{1}{\tilde{q}(z)} = \frac{1}{R(z)} - j \frac{\lambda}{\pi w(z)^2}
\]

\[
\frac{1}{\tilde{q}_0} = -j \frac{\lambda}{\pi w_0^2}
\]

Input wave is:

\[
\tilde{u}_0(x_0, y_0) = \sqrt{\frac{2}{\pi}} \frac{1}{w_0} e^{-j \frac{\pi}{\lambda} \frac{x_0^2 + y_0^2}{\tilde{q}_0}}
\]

Output wave becomes:

\[
\tilde{u}(\tilde{r}) = \sqrt{\frac{2}{\pi}} \frac{1}{w(z)} e^{-j \frac{\pi}{\lambda} \frac{x^2 + y^2}{\tilde{q}(z)}} e^{-j(kz-\phi)}
\]

where

\[
\phi(z) = \arctan \frac{\lambda z}{\pi w_0^2}
\]

\[
w(z) = w_0 \sqrt{1 + \left(\frac{\lambda z}{\pi w_0^2}\right)^2}
\]

\[
\tilde{q}(z) = \tilde{q}_0 + z
\]

The Gouy Phase Shift  Beam Spot Size  Propagation
A Gaussian Beam

$$\frac{1}{\tilde{q}(z)} = \frac{1}{R(z)} - j \frac{\lambda}{\pi w(z)^2}$$

$$\tilde{q}_0 = j \frac{\pi w_0^2}{\lambda}$$

$$\frac{1}{\tilde{q}_0} = - j \frac{\lambda}{\pi w_0^2}$$

$$q(z) = q(z_0) + (z - z_0)$$

z-direction

waist

2w₀

2√2w₀

zₐ

q(z) = q(z₀) + (z - z₀)
Propagating Gaussians

So if we start at a waist ($R \to \infty$):

$$\tilde{q}(z) = \tilde{q}_0 + z$$

Rewriting this,

$$\frac{1}{\tilde{q}(z)} = \frac{1}{\tilde{q}_0 + z}$$

Substituting for $q(z)$ and $q_0$ using definitions on previous pages, we get:

$$\frac{1}{R(z)} - j \frac{\lambda}{\pi w(z)^2} = \frac{z}{z^2 + \frac{\pi^2 w_0^4}{\lambda^2}} - \frac{j \pi w_0^2}{\left(z^2 + \frac{\pi^2 w_0^4}{\lambda^2}\right) \lambda}$$

Equating real and imaginary parts we get:

$$R(z) = z + \frac{\pi^2 w_0^4}{z \lambda^2}$$

$$w(z) = w_0 \sqrt{1 + \frac{z^2 \lambda^2}{\pi^2 w_0^4}}$$
So let's find the point the wave as expanded by factor $\sqrt{2}$:

So when $=1$, $w(z)$ is $\sqrt{2} \ w_0$

i.e.

$$z_R = \frac{\pi w_0^2}{\lambda}$$

The Rayleigh Range

Using this we can rewrite all of the previous expressions to be neater:
### Some Values for Gaussian Beams - 633nm

<table>
<thead>
<tr>
<th>$w_0$</th>
<th>$z_R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5mm</td>
<td>1.2m</td>
</tr>
<tr>
<td>1cm</td>
<td>500m</td>
</tr>
<tr>
<td>2μm</td>
<td>20μm</td>
</tr>
<tr>
<td>25cm</td>
<td>310km</td>
</tr>
</tbody>
</table>
Some more physical insight...

\[
R(z) = z + \frac{zR^2}{z}
\]

<table>
<thead>
<tr>
<th>For (z \ll z_R)</th>
<th>(R(z) \sim \text{large})</th>
<th>(R(z) = z)</th>
<th>(w(z) = w_0 \sqrt{1 + \left(\frac{z}{zR}\right)^2})</th>
</tr>
</thead>
<tbody>
<tr>
<td>For (z = z_R)</td>
<td>(R(z) = 2 z_R)</td>
<td>(R(z) = z_R)</td>
<td>(\sqrt{2} w_0)</td>
</tr>
<tr>
<td>For (z \gg z_r)</td>
<td>(R(z) = z)</td>
<td>(R(z) = z)</td>
<td>(w_0 \frac{z}{z_R} = \frac{\lambda z}{\pi w_0})</td>
</tr>
</tbody>
</table>

i.e. \(\theta \sim \frac{\lambda}{\pi w_0}\)
What does this mean?

\[ R(z_0) = z_0 \]
\[ R(z_1) = z_1 \]
\[ R(z_1) = R(z_0) + z_1 - z_0 \]

perhaps...

\[ q(z_1) = q(z_0) + z_1 - z_0 \]
More insight
Some limits

<table>
<thead>
<tr>
<th>Conditions</th>
<th>$R(z)$</th>
<th>Remarks</th>
<th>$w(z)$</th>
</tr>
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<tr>
<td>$z &lt;&lt; z_R$</td>
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Some Values for Gaussian Beams - 633nm

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<th>$\theta$</th>
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<tr>
<td>0.5mm</td>
<td>1.2m</td>
<td>0.4mrad</td>
</tr>
<tr>
<td>1cm</td>
<td>500m</td>
<td>20(\mu\text{rad})</td>
</tr>
<tr>
<td>2(\mu\text{m})</td>
<td>20(\mu\text{m})</td>
<td>100(\mu\text{rad (6°)})</td>
</tr>
<tr>
<td>25cm</td>
<td>310km</td>
<td>0.8(\mu\text{rad})</td>
</tr>
</tbody>
</table>
Another view of Gaussians

\[ \Delta x \Delta p > \hbar \]

If \( \Delta x \sim 2 w_0 \) and \( \theta \sim \lambda/(\pi w_0) \)

\[ \Delta p \sim \hbar k_x \sim \hbar k \theta \sim \hbar 2\pi/\lambda \lambda/(\pi w_0) \sim 2\hbar/w_0 \]
Summary of Properties

Near the waist: the beam intensity (amplitude) is near-constant, the wavefronts are near-planar (hence collimated beam). The G.B. here is essentially a plane wave with particular size.

Far from the waist: the wave is like a spherical wave with a defined size.

Near 1 R.R. from the waist: the beam radius is \( \sqrt{2} \) bigger than the waist, the on-axis intensity is a factor of 2 less than at the waist, radius of curvature is smallest and thus it is most curved (most divergent) - \( R \sim 2 Z_{rr} \).
What about that weird phase shift?
Using Huygens’ Integral

Start with a Gaussian plane wave i.e. \( R \Rightarrow \infty \)

\[
\frac{1}{\tilde{q}(z)} = \frac{1}{R(z)} - j \frac{\lambda}{\pi w(z)^2} \quad \frac{1}{\tilde{q}_0} = -j \frac{\lambda}{\pi w_0^2}
\]

Input wave is:

\[
\tilde{u}_0(x_0, y_0) = \sqrt{\frac{2}{\pi w_0}} e^{-j \frac{\pi}{\lambda} \frac{x_0^2 + y_0^2}{\tilde{q}_0}}
\]

Output wave becomes:

\[
\tilde{u}(\hat{r}) = \sqrt{\frac{2}{\pi w(z)}} e^{-j \frac{\pi}{\lambda} \frac{x^2 + y^2}{\tilde{q}(z)}} e^{-j(\kappa z - \phi)}
\]

where

\[
\phi(z) = \arctan \frac{\lambda z}{\pi w_0^2} \quad \omega(z) = w_0 \sqrt{1 + \left( \frac{\lambda z}{\pi w_0^2} \right)^2} \quad \tilde{q}(z) = \tilde{q}_0 + z
\]

The Gouy Phase  Beam Spot Size  Propagation
Observing the Gouy shift (1890)
Gouy Phase Shift

wavefronts of equal phase

lenses

$d = 2f$

Gouy phase shift

blanks
Paraxial Equation II

Assume $E(r,t)$ can be written as $E(r) e^{-j \omega t}$, then

$$\nabla^2 E + k^2 E = 0$$

(4.8)

where $k^2 = (\omega/c)^2 = (2\pi/\lambda)^2$ Helmholtz Equation

Suppress rapid $z$ fluctuations by...

$$E(r) = E_0(r) e^{-j k z}$$

If amplitude of residual wave is relatively slowly varying along the beam

$$|\lambda \frac{\partial E_0}{\partial z}| \ll |E_0| \quad \text{and} \quad |\lambda \frac{\partial^2 E_0}{\partial z^2}| \ll \left| \frac{\partial E_0}{\partial z} \right|$$

then we get:

$$(\nabla_T^2 - 2jk \frac{\partial}{\partial z}) E_0(r) = 0$$

(4.13)
The Gouy Phase shift of Focussed Beam
Physical origin of the Gouy phase shift

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We show explicitly that the well-known Gouy phase shift of any focused beam originates from transverse spatial confinement, which, through the uncertainty principle, introduces a spread in the transverse momenta and hence a shift in the expectation value of the axial propagation constant. A general expression is given for the Gouy phase shift in terms of expectation values of the squares of the transverse momenta. Our result also explains the phase shift in front of the Kirchhoff diffraction integral. © 2001 Optical Society of America